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CHAOTIC FEATURES OF THE GENERALIZED SHIFT MAP AND THE COMPLEMENTED SHIFT MAP

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Abstract

The aim of this paper is to prove some chaotic properties of the generalized shift map σ_n and the complemented shift map σ' such as chaotic dependence on initial conditions, topologically transitive**,** totally transitive. We have discussed some differences in the dynamics of the generalized and the complemented shift map with the shift map. Finally we have proved that the complemented shift map is chaotic.

Keywords**:** Symbolic dynamics, Shift map, Generalized shift map, Complemented shift map, Chaotic dependence on initial conditions, Totally transitive.

Introduction

The study of chaotic dynamics has become increasingly popular at the present days. Although there has been no universally accepted mathematical definition of chaos, it is generally believed that if for any system the distance between the nearby points increases and the distance between the far away point's decreases with time, the system is said to be chaotic. Chaotic dynamical systems constitute a special class of dynamical systems. Hence a dynamical system is chaotic if the orbit of it (or a subset of it) are confined to a bounded region, but still behave unpredictably. In 1975, Li and Yorke (1975) gave the first mathematical definition of chaos through the introduction of δ -scrambled set and in 1993, S. Li (1975) introduced the notion of ω - chaos through the introduction of ω -scrambled set. Devaney (1989) chaos is another popular type of chaos. Another interesting definition of chaos is generic chaos. In 2000, Murinova (1989) introduced generic chaos in metric spaces.

Devaney (1989) and Robinson (1999) both have given brilliant description of the space Σ_2. So by symbolic dynamical system we mean here the sequence space $\Sigma_2 = \{\alpha : \alpha = (\alpha_0 \alpha_1 \dots \dots \dots), \alpha_i = 0 \text{ or } 1\}$ along with the shift map defined on it. The points in this space will be infinite sequences of 0's and 1's.

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Symbolic dynamical systems (Σ_2, σ), (Σ_2, σ_n) and (Σ_2, σ') where Σ_2 is the sequence space, σ is the shift map, σ_n is the generalized shift map, and σ' is the complemented shift map are also examples of chaotic dynamical systems. In particular there are several works on symbolic dynamics where dynamics are represented by maps on symbol spaces.

The shift map obeys all the conditions of Devaney's definition (1989) of chaos such as sensitive dependence on initial conditions, chaotic dependence on initial conditions, topological transitivity and dense periodic points. Sensitive dependence on initial conditions is an important property for any chaotic map. There are some interesting research works on this particular property in (Du 1983, Du 2005, Robinsion 1999 and Glasner and Weiss 1993). Bau - Sen Du (1983) gave a new strong definition of chaos by using shift map in the symbol space Σ_2, and by taking a dense uncountable invariant scrambled set in Σ_2. We have proved that the shift map σ is generically δ –chaotic on Σ_2 with $\delta = 1$ by proving that σ is topologically mixing and hence it is weak mixing (Biswas, 2014).

In this paper, we discussed some special properties of the shift map. We proved some stronger chaotic properties of the generalized shift map and the complemented shift map. We also proved that the complemented shift map σ' is ω -chaotic with some additional features.

In theorem 1, theorem 4 and theorem 5, we have proved that the dynamical system (Σ_2, σ), (Σ_2, σ_n) and (Σ_2, σ') have chaotic dependence on initial conditions (Bhaumik and Choudhury, 2010). In theorem 2 and theorem 6, we have proved that the shift map σ and the complemented shift map σ' are totally transitive on Σ_2. In this paper, we have given a counter example to prove that not all topologically transitive maps are totally transitive. We also have given an example of a continuous function which is topologically transitive but not chaotic. We have discussed a comparison of the behaviors of the generalized shift map and the complemented shift map with the shift map which has been given in the last section.

Mathematical Preliminaries

In this section we give some definition and lemmas which are essential for the main results of this paper. We start with some elementary definitions.

Definition 1 (Shift map (Devaney, 1989**))**: The shift map $\sigma: \Sigma_2 \to \Sigma_2$ is defined by $\sigma(\alpha_0\alpha_1 \dots\dots\dots) = (\alpha_1\alpha_2 \dots\dots\dots)$, where $\alpha = (\alpha_0\alpha_1 \dots\dots\dots)$ is any point of Σ_2.

Definition 2 (Generalized shift map (Bhaumik and Choudhury, 2009)**):** The generalized shift map $\sigma_n: \Sigma_2 \to \Sigma_2$ is defined by $\sigma_n(s) = (s_n s_{n+1} s_{n+2} \dots\dots\dots)$, where $s = (s_0 s_1 \dots\dots\dots s_n \dots\dots)$ is any point of Σ_2. For $n = 1$, the generalized shift map reduces to the shift map and $n \geq 1$ is a finite positive integer.

Definition 3 (Complemented shift map (Bhaumik and Choudhury, 2010**)):** Let $s = (s_0 s_1 \ldots \ldots \ldots)$ be any point of Σ_2. Then the complemented shift map $\sigma' : \Sigma_2 \to \Sigma_2$ is defined by $\sigma'(s) = (s_1' s_2' \ldots \ldots \ldots \ldots)$, where $s_i'$is the complement of s_i. So it is the map which shifts the first element of a point and then also changes all others into its complement.

Definition 4 (Topologically Transitive ((Devaney, 1989)**))**: A continuous map $f: X \to X$ is called topologically transitive if for any pair of non empty open sets $U, V \subset X$ there exists $k \geq 0$ such that $f^k(U) \cap V \neq \phi$, where (X, d) is a compact metric space.

Definition 5 (Totally Transitive (Ruette, 2003**)):** Let (X , d) be a compact metric space. A continuous map $f : X \to X$ is called totally transitive if f^n is topologically transitive for all $n \geq 1$.

Definition 6 (Transitive Point ((Devaney, 1989)**):** Any point on a compact metric space (X, d) is called transitive point if it has dense orbit.

Definition 7 (Li -Yorke Pair (Li and Yorke, 1975**)):** A pair $(x, y) \in X^2$ is called Li - Yorke (with modulus $\delta > 0$) if $\underset{n\to\infty}{Lt\,sup}\, d\big(f^n(x), f^n(y)\big) \geq \delta$ and $\underset{n\to\infty}{Lt\,inf}\, d\big(f^n(x), f^n(y)\big) = 0$, where X is a compact metric space with the metric d and f is a continuous mapping on X. The set of all Li-Yorke pairs of modulus δ is denoted by $LY(f, \delta)$.

Definition 8 (Sensitive Dependence on initial Conditions (Devaney, 1989)**)**: A continuous map $f: X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of x , there exists $y \in N(x)$ and $n \geq 0$ such that $d\big(f^n(x), f^n(y)\big) > \delta$, where (X, d) is a compact metric space.

Definition 9 (**Chaotic dependence on initial conditions (**Blanchard et al., 2002**)):** A dynamical system (X, f) is called chaotic dependence on initial conditions if for any $x \in X$ and every neighborhood $N(x)$ of x there is a $y \in N(x)$ such that the pair $(x, y) \in X^2$ is Li-Yorke.

Definition 10 (Generically $\boldsymbol{\delta}$ −chaotic (Ruette, 2003**))**: Let $f: X \to X$ be a continuous map on a compact metric space X and $\delta > 0$. Then f is called generically δ −**chaotic** if $LY(f, \delta)$ is residual in X^2.

We also need the following lemmas.

Lemma 1 (Devaney, 1989)**:** Let $s, t \in \Sigma_2$and $s_i = t_i$, $i = 0,1, \ldots \ldots \ldots, m$. Then

$d(s, t) < \frac{1}{2^m}$ and conversely if $d(s, t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0,1, \ldots \ldots \ldots, m$.

Lemma 2 (Devaney, 1989)**:** Let X be a compact metric space and $T: X \rightarrow X$ is a continuous topologically mixing map then it is also (topologically) weak mixing map.

Lemma 3 (Ruette, 2003)**:** Let $T: X \rightarrow X$ be a continuous map on a compact metric space X. If T is (topologically) weak mixing then it is generically δ **−chaotic** on X with $\delta = diam(X)$.

Lemma 4 (Bhaumik and Choudhury, 2010)**:** The complemented shift map $\sigma': \Sigma_2 \rightarrow \Sigma_2$ is a continuous map on Σ_2.

Some special properties of the Shift Map

Property 1: The shift map $\sigma: \Sigma_2 \rightarrow \Sigma_2$ is generically δ −chaotic on Σ_2 with $\delta = 1$.

Proof: First we prove that the shift map is topologically mixing. Let U and V be two arbitrary non empty open set of Σ_2. Let, $u = (u_0 u_1 \dots \dots \dots) \epsilon U$ be any point such that $\min\{d(u, \gamma_1)\} = \varepsilon_1$, for any γ_1 belongs to the boundary of the set U and $v = (v_0 v_1 \dots \dots \dots) \in V$ be any point such that $\min\{d(v, \gamma_2)\} = \varepsilon_2$, for any γ_2 belongs to the boundary of the set V, where, $\varepsilon_1, \varepsilon_2 > 0$ be arbitrary.

We now choose two positive integers k_1 and k_2 such that $\frac{1}{2^{k_1 - 1}} < \Sigma_1$ and $\frac{1}{2^{k_2 - 1}} < \varepsilon_2$.

Next, we consider the sequence of points given by,

$\alpha_i = (u_0 u_1 \dots \dots u_{k_1 - 1} (1)^{i-1} v_0 v_1 \dots \dots v_{k_2 - 1} \dots \dots)$, for $i \geq 2$ and

$\alpha_1 = (u_0 u_1 \dots \dots u_{k_1 - 1} v_0 v_1 \dots \dots v_{k_2 - 1} \dots \dots)$.

We now prove the theorem with the help of lemma 1.

Now, $d(u, \alpha_i) < \frac{1}{2^{k_1 - 1}} < \varepsilon_1$, for all $i \geq 1$, (by lemma 1).

Hence, $\alpha_i \in U$, for all $i \geq 1$, that is $\sigma^k(\alpha_i) \in \sigma^k(U)$, for any $k \geq 0$ (1)

On the other hand, $\sigma^{k_1}(\alpha_1) = (v_0 v_1 \dots \dots v_{k_2 - 1} \dots \dots)$.

Hence, $d(\sigma^{k_1}(\alpha_1), v) < \frac{1}{2^{k_2 - 1}} < \varepsilon_2$,

by applying lemma 1 again. This gives $\sigma^{k_1}(\alpha_1) \in V$ also. (2)

By virtue of (1) and (2) we can say that $\sigma^{k_1}(U) \cap V \neq \phi$.

Next consider the point α_2. Then $\sigma^{k_1 + 1}(\alpha_2) = (v_0 v_1 \dots \dots v_{k_2 - 1} \dots \dots)$, which again belongs to V. Hence, $\sigma^{k_1 + 1}(U) \cap V \neq \phi$. Continuing this process by taking all α_i's we can easily prove that $\sigma^k(U) \cap V \neq \phi$, for all $k \geq k_1$.

Hence σ is topologically mixing on Σ_2. Since Σ_2 is a compact metric space and the shift map $\sigma: \Sigma_2 \to \Sigma_2$ is a continuous map, by lemma 2 it is also weak mixing. Again applying lemma 3 we get our desired result that is, $\sigma: \Sigma_2 \to \Sigma_2$is generically δ −chaotic on Σ_2 with $\delta = 1$.

Property 2: The shift map $\sigma: \Sigma_2 \to \Sigma_2$ has sensitive dependence on initial conditions.

Proof: We first prove that the shift map $\sigma: \Sigma_2 \to \Sigma_2$ is pointwise Lyapunovε-unstable on Σ_2. Let, $u = (u_0 u_1 \dots \dots \dots)$ be any point of Σ_2 and U be any open neighborhood of u. Since U is open we can always get an $\varepsilon > 0$ such that $\min\{d(u, \alpha)\} = \varepsilon$, for all α belonging to boundary of the set U. Now the maximum distance between any two points of Σ_2 is 1, by our chosen metric, so we cannot take $\varepsilon > 1$. Hence, $\varepsilon \le 1$ always, we take $n > 0$ such that $\frac{1}{2^n} < \varepsilon$. We now consider the point $v = (u_0 u_1 \dots \dots u_n u'_{n+1} u'_{n+2} u'_{n+3} \dots \dots)$ of Σ_2. Hence v is a point of Σ_2 which agrees with u up to u_n, but after the term u_n all the terms of v are complementary terms of that term of u.

Now, $d(u, v) < \frac{1}{2^n} < \in$, by lemma 1 and our construction above. Then obviously $v \in U$. Also, $d\big(\sigma^{n+1}(u), \sigma^{n+1}(v)\big) = d\big((u_{n+1} u_{n+2} \dots \dots), (u'_{n+1} u'_{n+2} \dots \dots)\big)$

$$= \frac{1}{2} + \frac{1}{2^2} + \dots \dots \dots$$

$$= 1.$$

So we can say that $d\big(\sigma^{n+1}(u), \sigma^{n+1}(v)\big) \ge \varepsilon$. That is the shift map $\sigma: \Sigma_2 \to \Sigma_2$ is Lyapunov ε-unstable at u. Hence the shift map $\sigma: \Sigma_2 \to \Sigma_2$ is point wise Lyapunovε-unstable on Σ_2. Again it is well known that the shift map is also topologically transitive. Since the shift map is both point wise Lyapunov ε-unstable and topologically transitive, it has sensitive dependence on initial conditions.

Property 3 (Bhaumik and Choudhury, 2010**):** The dynamical system (Σ_2, σ) has modified weakly chaotic dependence on initial conditions.

Chaotic dependence on initial conditions, Totally transitive, Topologically transitive

In this section, we try to prove that the dynamical systems $(\Sigma_2, \sigma), (\Sigma_2, \sigma_n)$ and (Σ_2, σ') have chaotic dependence on initial conditions. We consider a particular property of the dynamical system namely, the total transitivity. Here we show the chaotic map may or may not satisfy this property. In this paper, we also prove that the shift map and the complemented shift map are totally transitive on Σ_2. Here we give an example of a continuous function which is topologically transitive but not chaotic. We also give a suitable example illustrating that all topologically transitive maps are not totally transitive.

Theorem 1: The dynamical system (Σ_2, σ) has chaotic dependence on initial conditions.

Proof: Let, $p = (p_0 p_1 \dots \dots \dots)$ be any point of Σ_2. Also assume that U be any open neighborhood of p. Since, U is open we similarly take an $\varepsilon_2 > 0$, such that $\min\{d(p,\alpha)\} = \varepsilon_2$, for any α belongs to the boundary of the set U. Choose n sufficiently large such that $\frac{1}{2^n} < \varepsilon_2$. We now use some notations which help us to prove this theorem.

(i) Let Q= $(q_0 q_1 \dots \dots \dots . q_i)$ and $R = (r_0 r_1 \dots \dots \dots . r_m)$ are two finite sequences of 0's and 1's, then $QR = q_0 q_1 \dots \dots \dots . q_i r_0 r_1 \dots \dots \dots . r_m$. Further, if we suppose that $T_1, T_2, \dots \dots, T_r$ are r finite sequences of 0's and 1's, then $T_1 T_2 \dots \dots T_r$ can be defined in a similar manner as above.

(ii) Let $D(p, 2n+2) = (p'_{3n+1} p'_{3n+2} \dots \dots \dots p'_{4n+1} p_{4n+2} p_{4n+3} \dots \dots \dots p_{5n+2})$,

$D(p, 2n+4) = (p'_{5n+3} p'_{5n+4} \dots \dots \dots p'_{6n+4} p_{6n+5} p_{6n+6} \dots \dots \dots p_{7n+6})$,

$D(p, 2n+6) = (p'_{7n+7} p'_{7n+8} \dots \dots \dots p'_{8n+9} p_{8n+10} p_{8n+11} \dots \dots \dots p_{9n+12})$, and

so on.

(iii) Finally we take $t \in \Sigma_2$, such that,

$$t = (p_0 p_1 \dots \dots \dots . p_n (0)^n (1)^n D(p, 2n+2) D(p, 2n+4) D(p, 2n+6) \dots \dots \dots),$$

where $(\alpha)^n = \alpha\alpha \dots \dots \dots \dots \alpha$ n - times.

With those three notations and the lemma 1 above we now prove this theorem.

By construction p and t agree up to p_n. Hence, $d(p,t) < \frac{1}{2^n} < \varepsilon_2$, by lemma 1.

So, $t \in U$. Now, $\sigma^{3n+1}(p) = (p_{3n+1} p_{3n+2} \dots \dots \dots p_{4n+1} \dots \dots \dots)$ and

$\sigma^{3n+1}(t) = (p'_{3n+1} p'_{3n+2} \dots \dots . p'_{4n+1} \dots \dots \dots)$.

Note that *t* consists of infinitely many finite sequences of the type $D(p, 2n+k)$.

So we get,

$$\underset{n\to\infty}{Lt\,sup}\, d\big(\sigma^n(p), \sigma^n(t)\big)$$

$$\geq \underset{n\to\infty}{Lt}\, d((p_{3n+1} p_{3n+2} \dots \dots p_{4n+1} \dots \dots), (p'_{3n+1} p'_{3n+2} \dots \dots p'_{4n+1} \dots \dots))$$

$$\geq \underset{n\to\infty}{Lt} \left(\frac{1}{2} + \frac{1}{2^2} + \dots \dots \dots + \frac{1}{2^{n+1}}\right)$$

$=1$.

Hence, $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma^n(p), \sigma^n(t)\big) = 1.$ (3)

Similarly, $\sigma^{4n+2}(p) = (p_{4n+2}p_{4n+3} \dots \dots \dots p_{5n+2} \dots \dots)$ and

$\sigma^{4n+2}(t) = (p_{4n+2}p_{4n+3} \dots \dots \dots p_{5n+2} \dots \dots)$. So again we get,

$$\underset{n\to\infty}{Lt\ inf}\ d(\sigma^n(p), \sigma^n(t)))$$

$$\leq \underset{n\to\infty}{Lt}\ d((p_{4n+2}p_{4n+3} \dots \dots \dots p_{5n+2} \dots \dots ..), (p_{4n+2}p_{4n+3} \dots \dots \dots p_{5n+2} \dots \dots ..))$$

$$\leq \underset{n\to\infty}{Lt} (\frac{0}{2} + \frac{0}{2^2} + \dots \dots \dots .. + \frac{0}{2^{n+1}})$$

$$= 0.$$

Hence, $\underset{n\to\infty}{Lt\ inf}\ d(\sigma^n(p), \sigma^n(t)) = 0..$ (4)

From (3) and (4) it is proved that the pair (p, t) is Li-York. Hence, the dynamical system (Σ_2, σ) has chaotic dependence on initial conditions.

Hence, from property 3 and theorem 1 and by topological conjugacy discussed above we get our desired results.

Theorem 2 The shift map$\sigma: \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2.

Proof: Let U and V be two non empty open subsets of Σ_2 and $\in_1, \in_2 > 0$. Also let $p = (p_0p_1 \dots \dots \dots) \in U$ be a point such that $\min\{d(p, \beta_1)\} \geq \in_1$, for any β_1 belongs to the boundary of the set U. Similarly, let $t = (t_0t_1 \dots \dots \dots) \in V$ be any point such that $\min\{d(t, \beta_2)\} \geq \in_2$ for any β_2 belongs to the boundary of the set V. Next we choose integers k_1 and k_2 so large that $\frac{1}{2^{nk_1-1}} < \in_1$ and $\frac{1}{2^{nk_2}} < \in_2$, where n is any arbitrary positive integer. We now consider the point $\beta_3 = (p_0p_1 \dots \dots p_{nk_1-1}t_0t_1 \dots \dots t_{nk_2} \dots \dots)$. Then by lemma 1, $d(p, \beta_3) < \frac{1}{2^{nk_1-1}} < \in_1$.

Hence $\beta_3 \epsilon U$, that is $(\sigma^n)^{k_1}(\beta_3) \in (\sigma^n)^{k_1}(U)$.

On the other hand, $(\sigma^n)^{k_1}(\beta_3) = (t_0t_1 \dots \dots \dots t_{nk_2} \dots \dots)$.

Hence $((\sigma^n)^{k_1}(\beta_3), t) < \frac{1}{2^{nk_2}} < \in_2$, by applying lemma 1 again.

This gives $(\sigma^n)^{k_1}(\beta_3) \in V$.

Hence we get $(\sigma^n)^{k_1}(U) \cap V \neq \emptyset$. So the shift map $\sigma: \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2

Theorem 3 (Bhaumik and Choudhury, 2010**):** The generalized shift map (Σ_2, σ_n) is topologically mixing on Σ_2.

Theorem 4: The dynamical system (Σ_2, σ_n) has chaotic dependence on initial conditions.

Proof: At first we introduce some notations which help us to prove this theorem.

(i) Let $p = (p_0 p_1 \dots \dots \dots .)$ be any point of Σ_2 and U be any open neighborhood of p.

(ii) Let $P = (p_0 p_1 \dots \dots \dots . p_i)$ and Q= $(q_0 q_1 \dots \dots \dots . q_m)$ be two finite sequence of 0's and 1's, then $PQ = p_0 p_1 \dots \dots \dots . p_i q_0 q_1 \dots \dots \dots . q_m$. Further, if we suppose that $T_1, T_2, \dots \dots , T_p$ are p finite sequences of 0's and 1's; $T_1 T_2 \dots \dots T_p$ can be defined in a similar manner as above.

(iii) If β_i is any binary numeral, we denote the complement of β_i by β_i'. That is, if β_i =0 or 1, then β_i' =1 or 0.

(iv) Let $R_p(p, 2k+0) = (p'_{3nk} p'_{3nk+1} \dots \dots \dots p'_{4nk-1} p_{4nk} p_{4nk+1} \dots \dots \dots p_{5nk-1})$,

$R_p(p, 2k+2) = (p'_{5nk} p'_{5nk+1} \dots \dots \dots p'_{6nk+n-1} p_{6nk+n} p_{6nk+n+1} \dots \dots \dots p_{7nk+2n-1})$, and so on.

Note that for any even integer $m, R_p(p, 2k+m)$ is a finite string of length $(2nk + nm)$.

(v) Finally, we take $t \in \Sigma_2$ such that

$$t = \left(p_0 p_1 \dots \dots \dots p_{nk-1} (0)^{nk} (1)^{nk} R_p(p, 2k+0) R_p(p, 2k+2) R_p(p, 2k+4) \dots \dots \dots\right),$$

where $(\alpha)^{nk} = \alpha\alpha \dots \dots \dots \dots . \alpha$ nk-times.

We now consider the point p and the open neighborhood U of p defined in the above notation (i).

Since U is open we can always choose an $\varepsilon > 0$, such that $min\,\{d(p,\gamma)\} = \varepsilon$, for any γ belongs to the boundary of the set U. We choose k so large that $\frac{1}{2^{nk-1}} < \varepsilon$. By our construction p and t agree up to p_{nk-1}. Hence $d(p,t) < \frac{1}{2^{nk-1}} < \varepsilon$, by lemma 1. So $t \epsilon U$.

Now $\sigma_n^{3k}(p) = (p_{3nk} p_{3nk+1} \dots \dots p_{4nk-1} \dots \dots)$ and $\sigma_n^{3k}(t) = (p'_{3nk} p'_{3nk+1} \dots \dots p'_{4nk-1} \dots \dots)$.

Note that t consists of infinitely many finite sequences of the type $D(p, 2k+m)$.

So we get

$$\underset{k\to\infty}{Lt\,sup}\, d(\sigma_n^k(p), \sigma_n^k(t)) \geq \underset{k\to\infty}{Lt}\, d((p_{3nk} \dots \dots p_{4nk-1} \dots \dots), (p'_{3nk} \dots \dots p'_{4nk-1} \dots \dots))$$

$$\geq \underset{k\to\infty}{Lt} \left(\frac{1}{2} + \frac{1}{2^2} + \dots \dots \dots + \frac{1}{2^{nk}}\right)$$

$$=1.$$

Hence, $\underset{k\to\infty}{Lt\ sup}\ d\left(\sigma_n^k(s), \sigma_n^k(t)\right) = 1.$ (5)

Similarly,$\sigma_n^{4k}(p) = (p_{4nk}p_{4nk+1} \dots \dots p_{5nk-1} \dots \dots)$ and

$\sigma_n^{4k}(t) = (p_{4nk}p_{4nk+1} \dots \dots p_{5nk-1} \dots \dots).$

Again we get that

$$\underset{k\to\infty}{Lt\ inf}\ d(\sigma_n^k(p), \sigma_n^k(t)) \leq \underset{k\to\infty}{Lt}\ d((p_{4nk} \dots \dots p_{5nk-1} \dots \dots), (p_{4nk} \dots \dots p_{5nk-1} \dots \dots))$$
$$\leq \underset{k\to\infty}{Lt} (\tfrac{0}{2} + \tfrac{0}{2^2} + \ \dots \dots \dots .. + \tfrac{0}{2^{nk}})$$
$$= 0.$$

Hence, $\underset{k\to\infty}{Lt\ inf}\ d\left(\sigma_n^k(s), \sigma_n^k(t)\right) = 0.$ (6)

From (5) and (6) it is proved that the pair (p, t) is Li-York. Hence the dynamical system (Σ_2, σ_n) has chaotic dependence on initial conditions.

Theorem 5: The dynamical system (Σ_2, σ') has chaotic dependence on initial conditions.

Proof: Let $p = (p_0p_1 \dots \dots \dots .)$ be any point of Σ_2. Also let U be any open neighborhood of p. Since U is open, we can take an open ball $V \subset U$, with radius $\varepsilon > 0$. We choose n so large that $\frac{1}{2^n} < \varepsilon$. The following notations will assist us in proving this theorem.

(i) Let $P = (p_0p_1 \dots \dots \dots . p_i)$ and Q= $(q_0q_1 \dots \dots \dots . q_m)$be two finite sequence of 0's and 1's, then $PQ = p_0p_1 \dots \dots \dots . p_iq_0q_1 \dots \dots \dots . q_m$. Further, if we suppose that $T_1, T_2, \dots \dots, T_p$ are p finite sequences of 0's and 1's, then $T_1T_2 \dots \dots .. T_p$ can be defined in a similar manner as above.

(ii) If γ_q is any binary numeral, we denote the complement of γ_q by γ'_q, that is, if $\gamma_q = 0$ or $1, \gamma'_q = 1$ or 0.

(iii) Let $D(p, 2n + 2) = (p_{n+1}p_{n+2} \dots \dots \dots p_{2n+1}p'_{2n+2}p'_{2n+3} \dots \dots \dots p'_{3n+2})$,

$D(p, 2n + 4) = (p_{3n+3}p_{3n+4} \dots \dots \dots p_{4n+4}p'_{4n+5}p'_{4n+6} \dots \dots \dots p'_{5n+6})$,

$D(p, 2n + 6) = (p_{5n+7}p_{5n+8} \dots \dots p_{6n+9}p'_{6n+10}p'_{6n+11} \dots \dots \dots p'_{7n+12})$, and

so on.

Note that for any even integer $k > 0, D(p, 2n + k)$is a finite string of length $(2n + k)$.

(iv) Lastly, we take $t \in \Sigma_2$ such that,

$t = (p_0 p_1 \dots\dots\dots . p_n D(p, 2n+2) D(p, 2n+4) D(p, 2n+6) \dots\dots\dots\dots\dots\dots\dots)$.

With those four notations and lemma 1 as above we now prove the theorem. By construction p and t agree upto p_n. Hence $d(p,t) < \frac{1}{2^n} < \varepsilon$, by lemma 1. So $t \in V \Rightarrow t \in U$.

Now we consider the following two cases which are helpful to prove this theorem.

Case I: When n is an odd integer.

Here $\sigma'^{n+1}(p) = (p_{n+1} p_{n+2} \dots\dots\dots p_{2n+1} \dots\dots\dots)$ and

$\sigma'^{n+1}(t) = (p_{n+1} p_{n+2} \dots\dots\dots p_{2n+1} \dots\dots\dots)$.

Note that t consists of infinitely many finite sequences of the type $D(p, 2n+k)$. So we get,

$$\underset{n\to\infty}{Lt\ inf}\ d(\sigma'^n(p), \sigma'^n(t))$$

$$\leq \underset{n\to\infty}{Lt}\ d((p_{n+1} p_{n+2} \dots\dots\dots p_{2n+1} \dots\dots), (p_{n+1} p_{n+2} \dots\dots\dots p_{2n+1} \dots\dots))$$

$$\leq \underset{n\to\infty}{Lt} (\frac{0}{2} + \frac{0}{2^2} + \dots\dots\dots + \frac{0}{2^{n+1}})$$

$$= 0.$$

Hence, $\underset{n\to\infty}{Lt\ inf}\ d(\sigma'^n(p), \sigma'^n(t)) = 0.$ (7)

Similarly, $\sigma'^{2n+2}(p) = (p_{2n+2} p_{2n+3} \dots\dots\dots p_{3n+2} \dots\dots)$ and

$\sigma'^{2n+2}(t) = (p'_{2n+2} p'_{2n+3} \dots\dots\dots p'_{3n+2} \dots\dots)$.

So we get,

$$\underset{n\to\infty}{Lt\ sup}\ d(\sigma'^n(p), \sigma'^n(t))$$

$$\geq \underset{n\to\infty}{Lt}\ d((p_{2n+2} p_{2n+3} \dots\dots\dots p_{3n+2} \dots\dots), (p'_{2n+2} p'_{2n+3} \dots\dots\dots p'_{3n+2} \dots\dots))$$

$$\geq \underset{n\to\infty}{Lt} (\frac{1}{2} + \frac{1}{2^2} + \dots\dots\dots .. + \frac{1}{2^{n+1}})$$

$$=1.$$

Hence, $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma'^n(s), \sigma'^n(t)\big) = 1.$ (8)

Case II: When n is an even integer.

Then $\sigma'^{n+1}(p) = (p'_{n+1}p'_{n+2} \dots \dots \dots p'_{2n+1} \dots \dots)$ and

$\sigma'^{n+1}(t) = (p'_{n+1}p'_{n+2} \dots \dots \dots p'_{2n+1} \dots \dots)$.

In this case also t consists of infinitely many finite sequences of the type $D(p, 2n + k)$.

So we get,

$$\underset{n\to\infty}{Lt\ inf}\ d(\sigma'^{n}(p), \sigma'^{n}(t))$$

$$\leq \underset{n\to\infty}{Lt}\ d((p'_{n+1}p'_{n+2} \dots \dots \dots p'_{2n+1} \dots \dots), (p'_{n+1}p'_{n+2} \dots \dots \dots p'_{2n+1} \dots \dots))$$

$$\leq \underset{n\to\infty}{Lt} (\frac{0}{2} + \frac{0}{2^2} + \ \dots \dots \dots .. + \frac{0}{2^{n+1}})$$

$$= 0.$$

Hence, $\underset{n\to\infty}{Lt\ inf}\ d(\sigma'^{n}(p), \sigma'^{n}(t)) = 0.$ (9)

Similarly, $\sigma'^{2n+2}(p) = (p_{2n+2}p_{2n+3} \dots \dots \dots p_{3n+2} \dots \dots)$ and

$\sigma'^{2n+2}(t) = (p'_{2n+2}p'_{2n+3} \dots \dots \dots p'_{3n+2} \dots \dots)$.

So we get,

$$\underset{n\to\infty}{Lt\ sup}\ d(\sigma'^{n}(p), \sigma'^{n}(t))$$

$$\geq \underset{n\to\infty}{Lt}\ d((p_{2n+2}p_{2n+3} \dots \dots . p_{3n+2} \dots \dots), (p'_{2n+2}p'_{2n+3} \dots \dots . p'_{3n+2} \dots \dots))$$

$$\geq \underset{n\to\infty}{Lt} (\frac{1}{2} + \frac{1}{2^2} + \ \dots \dots \dots + \frac{1}{2^{n+1}})$$

$$=1.$$

Hence, $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma'^{n}(p), \sigma'^{n}(t)\big) = 1.$ (10)

By virtue of (7), (8), (9), (10) in the above two cases above we get that

$\underset{n\to\infty}{Lt\ inf}\ d(\sigma'^{n}(p), \sigma'^{n}(t)) = 0$ and $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma'^{n}(p), \sigma'^{n}(t)\big) = 1.$ (11)

By virtue of (11) it is proved that the pair (p, t) is Li-Yorke. Hence the dynamical system (Σ_2, σ') has chaotic dependence on initial conditions.

Theorem 6: The complemented shift map $\sigma' : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2.

Proof: We have to prove that σ'^n is topologically transitive (Bhaumik and Choudhury, 2009) for all $n \geq 1$. Let U and V be two non empty open subsets of Σ_2 and $\in_1, \in_2 > 0$. Also let $p = (p_0 p_1 \dots \dots \dots .) \in U$ be a point such that $\min\{d(p, \beta_1)\} \geq \in_1$, for any $\beta_1$belongs to the boundary of the set U . Similarly, let $t = (t_0 t_1 \dots \dots \dots) \in V$ be any point such that $\min\{d(p, \beta_2)\} \geq \in_2$for any β_2 belongs to the boundary of the set *V*. Next we choose two odd integers $k_1$and k_2 so large that $\frac{1}{2^{nk_1-1}} < \in_1$ and $\frac{1}{2^{nk_2}} < \in_2$.

Now we consider the following two cases which are helpful to prove this theorem.

Case I: When *n* is an even integer.

We now consider the point $\alpha = (p_0 p_1 \dots \dots p_{nk_1-1} t_0 t_1 \dots \dots t_{nk_2} \dots \dots)$. Then by lemma 1, $d(p, \alpha) < \frac{1}{2^{nk_1-1}} < \in_1$.

Hence $\alpha \epsilon U$, that is $(\sigma'^n)^{k_1}(\alpha) \in (\sigma'^n)^{k_1}(U)$.

On the other hand, $(\sigma'^n)^{k_1}(\alpha) = (t_0 t_1 \dots \dots t_{nk_2} \dots \dots)$.

Hence $d((\sigma'^n)^{k_1}(\alpha), t) < \frac{1}{2^{nk_2}} < \in_2$ by applying lemma 1 again.

This gives $(\sigma'^n)^{k_1}(\alpha) \in V$.

Hence we get $(\sigma'^n)^{k_1}(U) \cap V \neq \emptyset$, where *n* is any even integer. So the complemented shift map $\sigma' : \Sigma_2 \to \Sigma_2$ is totally transitive on $\Sigma_2$when *n* is an even integer.

Case II: When *n* is an odd integer.

In this case we consider the point $\beta = (p_0 p_1 \dots \dots p_{nk_1-1} t'_0 t'_1 \dots \dots t'_{nk_2} \dots \dots)$.

Thenby applying lemma 1 again. This gives $d(p, \beta) < \frac{1}{2^{nk_1-1}} < \in_1$. Hence $\beta \in U$,

that is$(\sigma'^n)^{k_1}(\beta) \in (\sigma'^n)^{k_1}(U)$. On the other hand, $(\sigma'^n)^{k_1}(\beta) = (t_0 t_1 \dots \dots t_{nk_2} \dots \dots)$.

Hence $d((\sigma'^n)^{k_1}(\beta), t) < \frac{1}{2^{nk_2}} < \in_2$ by applying lemma 1 again.

This gives $(\sigma'^n)^{k_1}(\beta) \in V$. Hence we get that $(\sigma'^n)^{k_1}(U) \cap V \neq \emptyset$, where *n* is any odd integer. So the complemented shift map $\sigma' : \Sigma_2 \to \Sigma_2$ is totally transitive on $\Sigma_2$when *n* is an odd integer.

Combining those two cases as above we get that the complemented shift map is totally transitive on Σ_2.

The chaotic maps are all topologically transitive and Li-Yorke sensitive. Now we try to give an example of a continuous function which is topologically transitive or Li-Yorke sensitive maps but not chaotic in the sense of B. S. Du.

Example 1: Let $h: [-1,1] \to [-1,1]$ be a function defined by

$$h(x) = \begin{cases} \frac{9}{8}(x+1), & -1 \le x \le -\frac{1}{9} \\ -9x, & -\frac{1}{9} \le x \le 0 \\ -x\,, & 0 \le x \le 1 \end{cases}$$

The function h defined above is obviously a continuous function. We can easily prove that the given function is topologically transitive and Li-Yorke sensitive. But it is not chaotic in the sense of B. S. Du since the closed interval [0,1] are jumping alternatively and never get close to each other.

We are also giving the following example showing that chaotic maps are not necessarily topologically transitive.

Example 2: Let $T(x) = 1 - |2x - 1|$for $0 \le x \le 1$ and let l be a continuous map from $\left[-\frac{1}{2}, 1\right]$ to itself defined by $l(x) = -x$ for$-\frac{1}{2} \le x \le 0$ and $l(x) = T(x)$ for $0 \le x \le 1$.

Then l is chaotic on $\left[-\frac{1}{2}, 1\right]$ but l is not topologically transitive.

Again from the following example, we can see that all topologically transitive maps are not totally transitive.

Example 3: Let $G(x)$ be a continuous map from [0,1] onto itself defined by

$$G(x) = \begin{cases} 4x + \frac{1}{5}, & 0 \le x \le \frac{1}{5} \\ -4x + \frac{9}{5}, & \frac{1}{5} \le x \le \frac{3}{5} \\ \frac{3}{2} - \frac{3}{2}x\,, & \frac{3}{5} \le x \le 1 \end{cases}$$

We can easily prove that the map G is topologically transitive on [0,1]. Here we can see that the subintervals $[0, \frac{3}{5}]$ and $[\frac{3}{5}, 1]$ are invariant under G^2, so G^2 is not topologically transitive on [0,1]. Hence it is not totally transitive. Therefore all topologically transitive maps are not totally transitive.

Problem 1: Show that the complemented shift map $\sigma': \Sigma_2 \to \Sigma_2$ is chaotic.

Solution: The complemented shift map $\sigma' : \Sigma_2 \to \Sigma_2$ is chaotic in the sense of Li- Yorke, because it is chaotic in the sense of Devaney. Because we know that Devaney chaos implies Li- Yorke. It can also be directly proved by the scrambled set *S* consisting of points, $\ulcorner_\alpha$ where $\ulcorner_\alpha$ is defined by,

$$\ulcorner_\alpha = \propto_0 01 \propto_0 \propto_0 \propto_1 \propto_1 0011 \propto_0 \propto_0 \propto_0 \propto_1 \propto_1 \propto_1 \propto_2 \propto_2 \propto_2 000111 \ldots \ldots \ldots \ldots,$$

for all $\alpha = (\alpha_0 \alpha_1 \ldots \ldots \ldots)$ in Σ_2. So this map is a strong chaotic map in the symbol space Σ_2. Since the shift map is often used to model the chaoticity of a dynamical system, we can now use the complemented shift map in place of the shift map to model the chaoticity of a dynamical system. So we conclude that the complemented shift map is a new model for chaotic dynamical systems.

Comparison of the generalized shift map and the complemented shift map with the shift map

In this section we discuss some basic differences of dynamics of the generalized shift map and the complemented shift map with the shift map. We also present a comparative study of the shift map with the generalized shift map and the complemented shift map.

We know that transitive points play a big role in any Devaney's chaotic system. For the shift map σ, if a point of Σ_2 which contains every finite sequence of 0's and 1's, the point is a transitive point. If we consider a point α of Σ_2 as given below,

$$\alpha = \left(\overbrace{(0)^n(1)^n}\,\overbrace{(00)^n(01)^n(10)^n(11)^n}\,\overbrace{(000)^n(001)^n} \ldots \ldots \ldots \overbrace{(0000)^n} \ldots \ldots)\right),$$

then obviously $\alpha \in \Sigma_2$ is a transitive point with respect to the generalized shift map. But

$\beta = \left(\overbrace{01}\,\overbrace{00\ 01\ 10\ 11}\,\overbrace{000\ 001} \ldots \ldots \overbrace{0000} \ldots \ldots \ldots)\right)$ is a transitive point for σ. Hence we can say that all transitive points of the generalized shift map σ_n are also transitive points of the shift map σ, but not conversely.

We now discuss the periodic points of the generalized shift map and the complemented shift map. Here the period means prime period. We start with the periodic points of those maps. If $\sigma : \Sigma_2 \to \Sigma_2$ is the shift map, we all know that any repeating sequence of 0's and 1's is always a periodic point of σ. For example, $c = (c_0 c_1 \ldots \ldots c_{n-1} c_0 c_1 \ldots \ldots c_{n-1} \ldots \ldots)$ is a periodic point of period *n* of σ, for all $n \geq 1$. But $\sigma_n(c) = c$, that is , c is a fixed point of σ_n. On the other hand if we consider the points (0000……….) and (1111………) of Σ_2. These are the only fixed points of σ. The above two points are fixed points of σ_n also, but there exist other fixed points of σ_n in Σ_2. Hence we conclude that periodic points of σ and σ_n are not same in general.

Again as for example, $a = (a_0a_1 \dots \dots a_m a_0 a_1 \dots \dots a_m \dots \dots)$is a periodic point of period m +1 of σ, for all $m \geq 0$. But in the case of the complemented shift map σ', the situation is different. In this case not all repeating sequences of 0'sand 1's are periodic in the same period that of σ. If u is a periodic point of σ such that it consists of repeating sequences of even number of terms, that is, $u = (u_0u_1 \dots \dots u_{2n-1}u_0u_1 \dots \dots u_{2n-1} \dots \dots)$ then obviously u is a periodic point of period $2n$ of σ'. On the other hand if we consider the point $v = (v_0v_1 \dots \dots v_{2n}v_0v_1 \dots \dots v_{2n} \dots \dots)$, it is a periodic point of $\sigma'$with period 2(2n +1), because

$$\sigma'^{2n+1}(v) = (v_1' \dots \dots v_{2n}'v_0'v_1' \dots \dots v_{2n}' \dots \dots) \neq v,$$

but $\sigma'^{2(2n+1)}(v) = (v_0v_1 \dots \dots v_{2n}v_0v_1 \dots \dots v_{2n} \dots \dots) = v$.

Hence for odd repeating sequences of 0's and 1's the case is different. So for an odd case we choose the point in a different manner. We now consider the point

$w = (w_0w_1 \dots \dots w_{2n}w_0'w_1' \dots \dots w_{2n}'w_0w_1 \dots \dots w_{2n}w_0'w_1' \dots \dots w_{2n}' \dots \dots)$, then

$$\sigma'^{2n+1}(w) = (w_0w_1 \dots \dots w_{2n}w_0'w_1' \dots \dots w_{2n}' \dots \dots \dots) = w.$$

So w is a periodic point of period 2n +1 of σ'. Similarly, we can get that w is a periodic point of σ with period 2(2n +1). Hence we conclude that the periodic points of σ and $\sigma'$are not same.

We know that one fixed point of any continuous map cannot be mapped into another fixed point by the map itself. We consider the points $\sigma_{f_1} = (0000 \dots \dots)$ and $\sigma_{f_2} = (1111 \dots \dots)$of Σ_2. These are the only fixed points of σ. Also we cannot jump from$\sigma_{f_1}$to σ_{f_2}(or from σ_{f_2} to σ_{f_1}) under any iteration of σ. But we can always jump from σ_{f_1} to σ_{f_2}(or from σ_{f_2} to σ_{f_1}) under iteration of σ'. Because $\sigma'(\sigma_{f_1}) = \sigma_{f_2}$and $\sigma'(\sigma_{f_2}) = \sigma_{f_1}$. Similarly, the points ${\sigma'}_{f_1} = (0101 \dots \dots.)$ and ${\sigma'}_{f_2} = (1010 \dots \dots.)$of Σ_2 are the only fixed points of σ'. Hence we cannot jump from ${\sigma'}_{f_1}$to ${\sigma'}_{f_2}$ (or from${\sigma'}_{f_2}$ to ${\sigma'}_{f_1}$) under any iteration of σ'. But $\sigma({\sigma'}_{f_1}) = {\sigma'}_{f_2}$ and $\sigma({\sigma'}_{f_2}) = {\sigma'}_1$. So we can say that the fixed points of the complemented shift map are mapped into one from another by the shift map. Similarly, the fixed points of the shift map are mapped into one from another by the complemented shift map.

Conclusions

In this article, we have proved some stronger chaotic properties of the generalized shift map and the complemented shift map. We also proved some special properties of the shift map and showed that in the symbol space the shift map is totally transitive. Hence we say that the shift map is the only map in the symbol space $\Sigma_2$which can be considered as a model of chaotic maps.

The generalized shift map has a property which is based on Li -Yorke pair but have some common features of sensitive dependence on initial conditions and it is topologically mixing on Σ_2 , which is a property stronger than topological transitivity.

In this paper we have proved that the complemented shift map is a ω -chaotic map. Since ω -chaos is equivalent to chaos in the sense of Devaney, the complemented shift map is Devaney chaotic. Hence it is also Li-Yorke chaotic, because Devaney chaos is stronger than Li-Yorke chaos. The property in Definition 9 is very important for any dynamical system, because this property is mainly based on Li-Yorke pair but has some common features with sensitive dependence on initial conditions. In this paper we have proved that the complemented shift map is totally transitive on Σ_2. Since the shift map is often used to model of the chaoticity of a dynamical system, we can now use the complemented shift map in place of the shift map to model of the chaoticity of a dynamical system. So we conclude that the complemented shift map is a new model for chaotic dynamical systems.

Although total transitivity is a stronger property than topological transitivity, every chaotic map does not necessarily become totally transitive. Hence, we conclude that in general not all transitive maps are totally transitive and also not all chaotic maps are totally transitive. In this article we discussed with examples that a continuous function which is topologically transitive but not chaotic in the sense of Du and all topologically transitive maps are not totally transitive.

In the last section we observed that all transitive points of the generalized shift map σ_n are also transitive points of the shift map σ, but not conversely and from this we conclude that periodic points of the shift map σ and the generalized shift map σ_n are not same in general. Similarly we showed that the periodic points of the shift map σ and the complemented shift map $\sigma'$are not same. We also proved that fixed points of the complemented shift map are mapped into one from another by the shift map and the fixed points of the shift map are mapped into one from another by the complemented shift map.

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