Barisal University Journal Part 1, 4(2):261-270 (2017) ISSN 2411-247X

INVESTIGATION OF CHAOTICITY OF THE GENERALIZED SHIFT MAP UNDER A NEW DEFINITION OF CHAOS AND COMPARE WITH SHIFT MAP

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Abstract

In this paper we have discussed the concept of the generalized shift map σ_n in the symbol space Σ_2. The aim of this paper is to investigate some chaotic properties of the generalized shift map σ_n such as chaotic dependence on initial conditions, topologically transitive**,** totally transitive. In this paper, we give a proof of the main theorem by constructing a dense uncountable invariant subset of the symbol space Σ_2 containing transitive points in a simpler way with the help of a different metric. !We have also discussed some differences in the dynamics of the generalized with the shift map. Finally we have also provided two examples, which support this new definition.

***Keywords*:** Symbolic Dynamics, Shift Map, Generalized shift map, Chaotic dependence on initial conditions, Totally transitive.

Introduction

The chaoticity of a dynamical system becomes a more demanding and challenging topic for both mathematicians and physicists. Li and Yorke (1975) are the first people that connect the term 'chaos' with a map. There are various types of chaotic maps, namely, tent map, quadratic map, logistic map etc. It is generally believed that if for a system the distance between the nearby points increases and the distance between the faraway points decreases with time, the system is said to be chaotic. Chaotic dynamical systems constitute a special class of dynamical systems. Hence a dynamical system is chaotic if the orbit of it (or a subset of it) are confined to a bounded region, but still behave unpredictably. In 1975, Li and Yorke (1975) gave the first mathematical definition of chaos through the introduction of δ -scrambled set. Devaney (1989) chaos is another popular type of chaos. Another interesting definition of chaos is generic chaos. In 2000, Murinova (1989) introduced generic chaos in metric spaces.

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According to Devaney (1989), a map f: $V \rightarrow V$ is said to be chaotic if the following three properties hold:

(i) f has sensitive dependence on initial conditions;

(ii) f is topologically transitive; and

(iii) periodic points are dense in V.

But in (iii), it is shown that topological transitivity and dense periodic orbits together imply sensitive dependence on initial conditions. So the condition (i) of Devaney's definition is redundant. It is also known that in an interval (not necessarily finite) a continuous, topological transitive map is chaotic in the sense of Devaney (1989). Akin introduced a linkage between the sensitivity and Li-Yorke (1975) version of chaos.

Devaney (1989) and Robinson (1999) both have given brilliant description of the space Σ_2. So by symbolic dynamical system we mean here the sequence space $\Sigma_2 = \{\alpha : \alpha = (\alpha_0 \alpha_1 \ldots\ldots\ldots), \alpha_i = 0 \text{ or } 1\}$ along with the shift map defined on it. The points in this space will be infinite sequences of 0's and 1's. In particular there are several works on symbolic dynamics where dynamics are represented by maps on symbol spaces.

The shift map obeys all the conditions of Devaney's definition (1989) of chaos such as sensitive dependence on initial conditions, chaotic dependence on initial conditions, topological transitivity and dense periodic points. Sensitive dependence on initial conditions is an important property for any chaotic map. There are some interesting research works on this particular property in (Du 1983, Du 2005, Robinsion 1999 and Glasner and Weiss 1993). Bau - Sen Du (1983) gave a new strong definition of chaos by using shift map in the symbol space Σ_2, and by taking a dense uncountable invariant scrambled set in Σ_2. We have proved that the shift map σ is generically δ –chaotic on Σ_2 with δ = 1 by proving that σ is topologically mixing and hence it is weak mixing (Biswas 2014).

In this paper, we introduced the concept of generalized shift map (Bhaumik and Choudhury, 2009). We discussed some stronger chaotic properties of the generalized shift map.

In the section 1 of this paper, we have given a counter example to prove that not all topologically transitive maps are totally transitive. We also have given an example of a continuous function which is topologically transitive but not chaotic. We discussed a comparison of the behaviors of the generalized shift map with the shift map which has been given in Section 2.

Definition 1 (**Chaotic dependence on initial conditions** (Blanchard et al., 2002)**):** A dynamical system (X, f) is called chaotic dependence on initial conditions if for any $x \in X$ and every neighborhood $N(x)$ of x there is a $y \in N(x)$ such that the pair $(x, y) \in X^2$ is Li-Yorke.

1. Results and Discussion

In this section, we try to prove that the dynamical systems (Σ_2, σ_n) has chaotic dependence on initial conditions. We consider a particular property of the dynamical system namely, the total transitivity. Here we show the chaotic map may or may not satisfy this property. In this paper, we also prove that the generalized shift map is totally transitive on Σ_2. We have to find that the generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions. Here we give an example of a continuous function which is topologically transitive but not chaotic. We also give a suitable example illustrating that all topologically transitive maps are not totally transitive.

Lemma 1 (Devaney 1989)**:** Let $s, t \in \Sigma_2$ and $s_i = t_i$, $i = 0,1, \ldots\ldots\ldots, m$. Then $d(s, t) < \frac{1}{2^m}$ and conversely if $d(s, t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0,1, \ldots\ldots\ldots, m$.

Lemma 2 (Devaney 1989)**:** Let X be a compact metric space and $T: X \to X$ is a continuous topologically mixing map then it is also (topologically) weak mixing map.

Theorem 1: (Bhaumik and Choudhury 2010**):** The generalized shift map $(\Sigma_2, \sigma_n)$is topologically mixing on Σ_2.

Theorem 2: The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has chaotic dependence on initial conditions.

Proof: At first we introduce some notations which help us to prove this theorem.

(i) Let $p = (p_0 p_1 \ldots\ldots\ldots)$ be any point of Σ_2 and U be any open neighborhood of p.

(ii) Let $P = (p_0 p_1 \ldots\ldots\ldots p_l)$ and Q= $(q_0 q_1 \ldots\ldots\ldots q_m)$ be two finite sequence of 0's and 1's, then $PQ = p_0 p_1 \ldots\ldots\ldots p_l q_0 q_1 \ldots\ldots\ldots q_m$. Further, if we suppose that $T_1, T_2, \ldots\ldots, T_p$ are p finite sequences of 0's and 1's; $T_1 T_2 \ldots\ldots T_p$can be defined in a similar manner as above.

(iii) If β_i is any binary numeral, we denote the complement of β_i by β_i'. That is, if β_i =0 or 1, then β_i' =1 or 0.

(iv) Let $R_p(p, 2k + 0) = (p'_{3nk} p'_{3nk+1} \ldots\ldots\ldots p'_{4nk-1} p_{4nk} p_{4nk+1} \ldots\ldots\ldots p_{5nk-1})$.

$R_p(p, 2k+2) = (p'_{6nk}p'_{6nk+1} \dots\dots\dots p'_{6nk+n-1}p_{6nk+n}p_{6nk+n+1} \dots\dots\dots p_{7nk+2n-1})$, and so on.

Note that for any even integer $m, R_p(p, 2k+m)$ is a finite string of length $(2nk+nm)$.

(v) Finally, we take $t \in \Sigma_2$ such that

$t = (p_0p_1 \dots\dots\dots p_{nk-1}(0)^{nk}(1)^{nk}R_p(p, 2k+0)R_p(p, 2k+2)R_p(p, 2k+4) \dots\dots\dots)$,

where $(\alpha)^{nk} = \alpha\alpha \dots\dots\dots\dots \alpha$ nk-times.

We now consider the point p and the open neighborhood U of p defined in the above notation.

(i) Since U is open we can always choose an $\varepsilon > 0$, such that $min\{d(p,\gamma)\} = \varepsilon$, for any γ belongs to the boundary of the set U. We choose k so large that $\frac{1}{2^{nk-1}} < \varepsilon$. By our construction p and t agree up to p_{nk-1}. Hence $d(p,t) \le \frac{1}{2^{nk-1}} < \varepsilon$, by lemma 1. So $t \in U$.

Now $\sigma_n^{3k}(p) = (p_{3nk}p_{3nk+1} \dots\dots p_{4nk-1} \dots\dots)$ and $\sigma_n^{3k}(t) = (p'_{3nk}p'_{3nk+1} \dots\dots p'_{4nk-1} \dots\dots)$

Note that t consists of infinitely many sequences of the type $D(p, 2k+m)$.

So we get

$$\underset{k\to\infty}{Lt\,sup}\, d(\sigma_n^k(p), \sigma_n^k(t)) \ge \underset{k\to\infty}{Lt}\, d((p_{3nk} \dots\dots p_{4nk-1} \dots\dots), (p'_{3nk} \dots\dots p'_{4nk-1} \dots\dots))$$

$$\ge \underset{k\to\infty}{Lt}\left(\frac{1}{2} + \frac{1}{2^2} + \dots\dots\dots + \frac{1}{2^{nk}}\right)$$

$$=1.$$

Hence, $\underset{k\to\infty}{Lt\,sup}\, d\left(\sigma_n^k(s), \sigma_n^k(t)\right) = 1$, (1.1)

Similarly, $\sigma_n^{4k}(p) = (p_{4nk}p_{4nk+1} \dots\dots p_{5nk-1} \dots\dots)$ and

$\sigma_n^{4k}(t) = (p_{4nk}p_{4nk+1} \dots\dots p_{5nk-1} \dots\dots)$.

Again we get that

$$\underset{k\to\infty}{Lt\,inf}\, d(\sigma_n^k(p), \sigma_n^k(t)) \le \underset{k\to\infty}{Lt}\, d((p_{4nk} \dots\dots p_{5nk-1} \dots\dots), (p_{4nk} \dots\dots p_{5nk-1} \dots\dots))$$

$$\le \underset{k\to\infty}{Lt}\left(\frac{0}{2} + \frac{0}{2^2} + \dots\dots\dots + \frac{0}{2^{nk}}\right)$$

$$= 0.$$

Hence $\underset{k\to\infty}{Lt\,inf}\, d\left(\sigma_n^k(s), \sigma_n^k(t)\right) = 0$. (1.2)

From (1.1) and (1.2) it is proved that the pair (p,t) is Li-York. Hence the dynamical system (Σ_2, σ_n) has chaotic dependence on initial conditions.

Theorem 3: The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.

Proof: Let $x = (x_0 x_1 \dots\dots)$ be any point of Σ_2 and U be any non empty open set of Σ_2. Hence we can take an open ball V with radius $\in > 0$ and center at $= (\alpha_0 \alpha_1 \dots \dots)$, such that $V \subset U$. Let $p > 0$ be an integer such that $\frac{1}{2^{np-1}} < \in$. We now consider the point $y = (\alpha_0 \alpha_1 \dots \dots \alpha_{np-1} x'_{np} x'_{np+1} \dots\dots)$. Then the point y agrees with α up to α_{np-1} and after that all terms of y are the complementary terms of the point x starting with x'_{np}.

By the application of lemma 1 above we get that $d(\alpha, y) < \frac{1}{2^{np-1}} < \in$. Hence $y \in V$, that is $y \in U$ also.

Again we get $d\left(\sigma_n^p(x), \sigma_n^p(y)\right) = d\left((x_{np} x_{np+1} \dots\dots), (x'_{np} x'_{np+1} \dots\dots)\right)$

$$= \frac{1}{2} + \frac{1}{2^2} + \cdots$$

$$= 1$$

that is $y \in U$ such that $d\left(\sigma_n^p(x), \sigma_n^p(y)\right) = 1$, where U is an arbitrary open set of Σ_2.

Hence the generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.

If a continuous map has strong sensitive dependence on initial conditions then it has sensitive dependence on initial conditions but the converse is not always true. The following example establishes this fact.

Example 1: Let $k : [-1,1] \to [-1,1]$ be a map defined by

$$k(x) = \begin{cases} \frac{8}{7}x + \frac{8}{7}, & -1 \le x \le -\frac{1}{8} \\ -8x, & -\frac{1}{8} \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$

The function k defined above is obviously a continuous map. Also it can easily proved that the function has sensitive dependence on initial conditions. We observe that maximum distance between any two points of [-1, 1] is equal to 2. We consider the point $-\frac{9}{16}$ and the open interval $U = (0,1)$. Then there exists no point $y \in U$ such that

$d(k^n(x), k^n(y)) = 2$, for any $n \geq 0$. Hence $k(x)$ does not strong sensitive dependence on initial conditions.

Theorem 4: The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2.

Proof: We have to prove that σ_n is topologically transitive (Bhaumik and Choudhury, 2009) for all

$n \geq 1$. Let U and V be two non empty open subsets of Σ_2 and $\epsilon_1, \epsilon_2 > 0$. Also let $p = (p_0 p_1 \ldots\ldots\ldots) \in U$ be a point such that $\min\{d(p, \beta_1)\} \geq \epsilon_1$, for any β_1 belongs to the boundary of the set U. Similarly, let $t = (t_0 t_1 \ldots\ldots\ldots) \in V$ be any point such that $\min\{d(p, \beta_2)\} \geq \epsilon_2$ for any β_2 belongs to the boundary of the set V. Next we choose two odd integers k_1 and k_2 so large that $\frac{1}{2^{nk_1 - 1}} < \epsilon_1$ and $\frac{1}{2^{nk_2}} < \epsilon_2$.

Now we consider the following two cases which are helpful to prove this theorem.

Case I: When n is an even integer.

We now consider the point $\alpha = (p_0 p_1 \ldots\ldots p_{nk_1 - 1} t_0 t_1 \ldots\ldots t_{nk_2} \ldots\ldots)$. Then by lemma 1, $d(p, \alpha) < \frac{1}{2^{nk_1 - 1}} < \epsilon_1$.

Hence $\alpha \epsilon U$, that is $(\sigma_n)^{k_1}(\alpha) \in (\sigma_n)^{k_1}(U)$.

On the other hand, $(\sigma_n)^{k_1}(\alpha) = (t_0 t_1 \ldots\ldots t_{nk_2} \ldots\ldots)$.

Hence $d((\sigma_n)^{k_1}(\alpha), t) < \frac{1}{2^{n k_2}} < \epsilon_2$ by applying lemma 1 again.

This gives $(\sigma_n)^{k_1}(\alpha) \in V$.

Hence we get $(\sigma_n)^{k_1}(U) \cap V \neq \emptyset$, where n is any even integer.

So the generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2 when n is an even integer.

Case II: When n is an odd integer.

In this case we consider the point $\beta = (p_0 p_1 \ldots\ldots p_{nk_1 - 1} t'_0 t'_1 \ldots\ldots t'_{nk_2} \ldots\ldots)$.

Then by applying lemma 1 again. This gives $d(p, \beta) < \frac{1}{2^{nk_1 - 1}} < \epsilon_1$. Hence $\beta \in U$,

that is $(\sigma_n)^{k_1}(\beta) \in (\sigma_n)^{k_1}(U)$. On the other hand, $(\sigma_n)^{k_1}(\beta) = (t_0 t_1 \ldots\ldots t_{nk_2} \ldots\ldots)$.

Hence $d((\sigma_n)^{k_1}(\beta), t) < \frac{1}{2^{nk_2}} < \epsilon_2$ by applying lemma 1 again.

This gives $(\sigma_n)^{k_n}(\beta) \in V$. Hence we get that $(\sigma_n)^{k_n}(U) \cap V \neq \phi$, where n is any odd integer. So the generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2 when n is an odd integer.

Combining those two cases as above we get that the generalized shift map is totally transitive on Σ_2.

The chaotic maps are all topologically transitive and Li-Yorke sensitive. Now we try to give an example of a continuous function which is topologically transitive or Li-Yorke sensitive maps but not chaotic in the sense of Du (1998)

We are also giving the following example showing that chaotic maps are not necessarily topologically transitive.

Example 2: Let $T(x) = 1 - |2x - 1|$ for $0 \le x \le 1$ and let d_1 be a continuous map from $\left[-\frac{1}{2}, 1\right]$ to itself defined by $d_1(x) = -x$ for $-\frac{1}{2} \le x \le 0$ and $d_1(x) = T(x)$ for $0 \le x \le 1$. Then d_1 is chaotic on $\left[-\frac{1}{2}, 1\right]$ but d_1 is not topologically transitive.

Again from the following example, we can see that all topologically transitive maps are not totally transitive.

Example 3: Let $J(x)$ be a continuous map from [0,1] onto itself defined by

$$J(x) = \begin{cases} 4x + \frac{1}{2}, & 0 \le x \le \frac{1}{8} \\ -4x + \frac{9}{8}, & \frac{1}{8} \le x \le \frac{3}{8} \\ \frac{3}{2} - \frac{3}{2}x, & \frac{3}{8} \le x \le 1 \end{cases}$$

We can easily prove that the map J is topologically transitive on [0,1]. Here we can see that the subintervals $[0, \frac{3}{5}]$ and $[\frac{3}{5}, 1]$ are invariant under J^2, so J^2 is not topologically transitive on [0,1]. Hence $J(x)$ is not totally transitive on [0,1]. Therefore all topologically transitive maps are not totally transitive.

Comparison of the generalized shift map with the shift map

In this section we discuss some basic differences of dynamics of the generalized shift map with the shift map. We also present a comparative study of the shift map with the generalized shift map.

We know that transitive points play a big role in any Devaney's chaotic system. For the shift map σ, if a point of Σ_2 which contains every finite sequence of 0's and 1's, the point is a transitive point. If we consider a point α of Σ_2 as given below,

$$\alpha = \left(\overline{(0)^n(1)^n}\quad\overline{(00)^n(01)^n(10)^n(11)^n}\quad\overline{(000)^n(001)^n}\ldots\ldots\overline{(0000)^n}\ldots\ldots\right),$$

then obviously $\alpha \in \Sigma_2$ is a transitive point with respect to the generalized shift map. But

$\beta = \left(\overline{01}\quad\overline{00\ 01\ 10\ 11}\quad\overline{000\ 001}\ldots\ldots\overline{0000}\ldots\ldots\right)$ is a transitive point for σ. Hence we can say that all transitive points of the generalized shift map σ_n are also transitive points of the shift map σ, but not conversely.

We now discuss the periodic points of the generalized shift map. Here the period means prime period. We start with the periodic points of those maps. If $\sigma:\Sigma_2 \to \Sigma_2$ is the shift map, we all know that any repeating sequence of 0's and 1's is always a periodic point of σ. For example, $c = (c_0c_1\ldots\ldots c_{n-1}c_0c_1\ldots\ldots c_{n-1}\ldots\ldots)$ is a periodic point of period *n* of σ, for all $n \geq 1$. But $\sigma_n(c) = c$, that is, *c* is a fixed point of σ_n. On the other hand if we consider the points (0000……….) and (1111………) of Σ_2. These are the only fixed points of σ. The above two points are fixed points of σ_n also, but there exist other fixed points of σ_n in Σ_2. Hence we conclude that periodic points of σ and σ_n are not same in general.

Conclusions

In this article, we have proved some stronger chaotic properties of the generalized shift map. The generalized shift map has a property which is based on Li -Yorke pair but have some common features of sensitive dependence on initial conditions and it is topologically mixing on Σ_2 , which is a property stronger than topological transitivity.

Since the generalized shift map is chaotic in the sense of Devaney, it is also Li-Yorke chaotic, because Devaney chaos is stronger than Li-Yorke chaos. The property in Definition 1 is very important for any dynamical system, because this property is mainly based on Li-Yorke pair but has some common features with sensitive dependence on initial conditions. In this paper we have proved that generalized shift map has strong sensitive dependence on initial conditions. Since the shift map is often used to model of the chaoticity of a dynamical system, we can now try to use the generalized shift map in place of the shift map to model of the chaoticity of a dynamical system. So we conclude that the generalized shift map is a new model for chaotic dynamical systems.

Although total transitivity is a stronger property than topological transitivity, every chaotic map does not necessarily become totally transitive. Hence, we conclude that in general not all transitive maps are totally transitive and also not all chaotic maps are totally transitive. In this article we discussed with examples that a continuous function which is topologically transitive but not chaotic in the sense of Du and all topologically transitive maps are not totally transitive.

In the last section we observed that all transitive points of the generalized shift map σ_n are also transitive points of the shift map σ, but not conversely and from this we conclude that periodic points of the shift map σ and the generalized shift map σ_n are not same in general.

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