Barishal University Journal Part 1, 5(1&2): 47-58 (2018) ISSN 2411-247X

A STUDY OF SMOOTH MAP ON MANIFOLDS

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Abstract

In this paper we present a theorem to determine characteristics of smooth map. For this purpose we studied some propositions and lemma related to smooth map on manifolds. The basic geometry of tangent space and definitions involving tangent bundle are discussed in our paper. We developed the notion of quotient manifold and sub manifold using the concept of smooth map. Finally, the theorem 6.1 on smooth map is generalized for finding the neighborhoods of surjective map.

Keywords: Tangent space, Smooth map, Tangent bundle, Diffeomorphism, Manifolds.

1. Introduction

A manifold is a topological space that locally resembles Euclidean space near each point. Although a manifold locally resembles Euclidean space, meaning that every point has a neighborhoods homeomorphism to an open subset of Euclidean space. The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows complicated structures to be described and understood in terms of the simpler local topological properties of Euclidean space. Manifolds naturally arise as a solution set of a system of equation and graphs of functions. Manifolds can be equipped with additional structure. One important class of manifolds is the class of differentiable manifolds. This differentiable manifolds structure allows calculus to be done on manifolds. The study of manifolds combines many important areas of mathematics. It generalizes concept such as curves and surfaces as well as ideas from linear algebra and topology. There were several important results of manifolds between 18th and 19th century mathematics. The oldest of these was Non-Euclidean geometry, which considers spaces where Euclids parallel postulate fails. The Italian mathematician Saccheri (1733)

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first studied geometry and then Lobachevsky and Bolyai (1830) developed it. Their research uncovered two types of spaces whose geometric structures differ from that of classical Euclidean space. These are called hyperbolic geometry and elliptic geometry. In the modern theory of manifolds, these notions correspond to manifolds with constant, negative and positive curvature respectively. Gauss (1855) is the first to consider abstract spaces as mathematical objects in their own right. His theorem egregium gives a method for computing the curvature of a surface without considering the ambient space in which surface lies. Manifold theory has come to focus exclusively on these intrinsic properties while largely ignoring the extrinsic properties of the ambient space. Gauss (1805) and Monge (1807) first introduced differential geometry. The important contributions were made by many scientists in 19th century. Darboux and Bianchi (1896) collected and systematized the work. However, recently there are several researchers who worked on the development of several parts of manifolds. Ahmed et al., (2012) developed the characterization of vector fields on manifolds, Ali et al., (2012) worked on the exterior algebra with differential forms on manifolds. Ahmed et al., (2014) introduced the multi linear algebra and tensors with vector sub bundle on manifolds, Osman (2016) worked on basic integration on smooth manifolds and application map with Stokes theorem. In this paper we shall discuss the properties of tangent space, tangent bundle and developed the notion of quotient manifold and sub manifolds using the concept of smooth map. In this paper some necessary propositions related to smooth map are treated and the theorem 6.1 has been derived.

2. Tangent spaces

Definition 2.1 (Flanders. 1963**)**: For embedded sub manifolds $M \subseteq \mathbb{R}^n$, the tangent space $T_a(M)$ at $a \in M$ can be defined as the set of all velocity vectors $v = \dot{\gamma}(0)$, where $\gamma: J \to M$ is a smooth curve with $\gamma(0) = a$, here $j \subseteq \mathbb{R}$ is an open interval around 0.

It turns out that $T_a(M)$ becomes a vector subspace of $\mathbb{R}^n$.

Example 2.1 (Flanders. 1963**)** Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of x such that $\| x \|^2 = 1$. A curve $\gamma(t)$ lies in S^n if and only if $\| \gamma(t) \| = 1$. Taking the derivative of the equation $\gamma(t).\gamma(t) = 1$ at $t = 0$. We obtain after dividing by 2 and using $\gamma(0) = a$, $a\dot{\gamma}(0) = 0$. That is $T_a(M)$ consist of vectors $v \in \mathbb{R}^{n+1}$ that are orthogonal to $a \in \mathbb{R}^3\{0\}$. It is easily seen that every such vector v is of the form $\dot{\gamma}(0)$, hence that $T_a{}^{S^n} = (\mathbb{R}p)^\perp$, hence the hyperplane orthogonal to the line through a.

Definition 2.2 : Let M be a manifold and $a \in M$. The tangent space $T_a(M)$ is the set of all linear maps $v: C^\infty(M) \to \mathbb{R}$ of the form

$$v(f) = \frac{d}{dt}|_{t=0}\, f(\gamma(t))$$

For smooth curve $\gamma \in C^{\infty}(J, M)$ with $\gamma(0) = a$. The elements $v \in T_a(M)$ are called the tangent vectors to M at a.

The following local coordinate description makes it clear that $T_a(M)$ is linear subspace of the vector space $L(C^{\infty}(M), \mathbb{R})$ of linear maps $C^{\infty}(M) \to \mathbb{R}$. The dimension of linear maps equal to the dimension of M.

Theorem 2.1 (Narasimhan. 1968) : Let (U, φ) be a coordinate chart around a. A linear map $v: C^{\infty}(M) \to \mathbb{R}$ is in $T_a(M)$ if and only if it has the form

$$v(f) = \sum_{i=1}^{m} p^i \frac{\partial(f o \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(a)} \tag{2.1}$$

for some $p = (p^1, \dots, p^m) \in \mathbb{R}^n$.

Proof. Given a linear map v of this form. Let $\widetilde{\gamma}: \mathbb{R} \to \varphi(U)$ be a curve with $\widetilde{\gamma}(t) = \varphi(a) + tp$ for $|t|$ sufficiently small. Let $\gamma = \varphi^{-1} o\, \widetilde{\gamma}$. Then

$$\frac{d}{dt}\Big|_{t=0} f\big(\gamma(t)\big) = \frac{d}{dt}\Big|_{t=0} (f o \varphi^{-1})(\,\varphi(a) + tp)$$

$$= \sum_{i=1}^{m} p^i \frac{\partial(f o \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(a)} \tag{2.2}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = a$, let $\widetilde{\gamma} = \varphi\, o\, \gamma$ be the corresponding curve in $\varphi(U)$. Then $\widetilde{\gamma} = \varphi(a)$ and

$$\frac{d}{dt}\Big|_{t=0} f\big(\gamma(t)\big) = \frac{d}{dt}\Big|_{t=0} (f o \varphi^{-1})(\widetilde{\gamma}(t)\,)$$

$$= \sum_{i=1}^{m} p^i \frac{\partial(f o \varphi^{-1})}{\partial u^i}\Big|_{u=\gamma(a)}, \tag{2.3}$$

where $a = \frac{d\widetilde{\gamma}}{dt}\Big|_{t=0}$. We can use this result as an alternative definition of the tangent space.

Definition 2.3 (Brickell et al., 1970) : Let M be a smooth manifold, and let $\mathcal{Y}(M)$ be the ring of smooth functions on M.

A tangent space of M at a point $a \in M$ is a linear map $\xi: \mathcal{Y}(M) \to \mathbb{R}$ such that

$$\xi(fg) = \xi(f)\, g(a) + f(a)\, \xi(g), \quad f, g \in \mathcal{Y}(M).$$

The tangent vectors form a real vector space, $T_a(M)$, under the linear operations

$$(\lambda\xi + \mu\eta)(f) = \lambda\xi(f) + \mu\eta(f), \quad \lambda, \mu \in \mathbb{R}, \quad \xi, \eta \in T_a(M), f \in \mathcal{Y}(M) \tag{2.4}$$

$T_a(M)$ is called the tangent space of M at a.

3. Smooth functions on manifolds

A real-valued function on an open subset $M \subseteq \mathbb{R}^n$ is called smooth if it is infinitely differentiable. The notion of smooth functions on open subsets of Euclidean spaces carries over to manifolds. A function is smooth if its expression in local coordinates is smooth.

Definition 3.1 (Brickell et al., 1970) : A function $f: M \to \mathbb{R}$ on a manifold M is called smooth if for all charts (U, φ) of the function

$$fo\varphi^{-1}: \varphi(U) \to \mathbb{R} \tag{3.1}$$

is smooth. The set of smooth functions on M is denoted by $C^\infty(M)$.

Example 3.1 The height function $f: S^2 \to \mathbb{R}, (x, y, z) \to z$ is smooth. In fact, we see that for any smooth function $h \in C^\infty(\mathbb{R}^3)$, the restriction $f = h|_{S^2}$ is again smooth.

Lemma 3.1 (Olum. 1953) : Smooth functions $f \in C^\infty(M)$ are continuous.

Proof. For every open subset $j \subseteq \mathbb{R}$, the pre-image $f^{-1}(J) \subseteq M$ is open. We have to show that for every (U, φ), the set $\varphi(U \cap f^{-1}(J)) \subseteq \mathbb{R}^n$ is open. But this subset coincides with the pre-image of J under the map $fo\varphi^{-1}: \varphi(U) \to \mathbb{R}$, which is a smooth function on an open subset of $\mathbb{R}^n$ and these are continuous.

4. The derivative of a smooth map

Let $\varphi: M \to N$ be a smooth map. Recall that φ induces a homomorphism

$$\varphi^*: \mathcal{Y}(M) \leftarrow \mathcal{Y}(N)$$

given by

$$(\varphi^* f)(x) = f\big(\varphi(x)\big), \quad f \in \mathcal{Y}(N), \quad x \in M. \tag{4.1}$$

Lemma 4.1 (Narasimhan. 1968) : Let $\xi \in T_a(M)$. Then $\xi \circ \varphi^* \in T_{\varphi(a)}(N)$, and the correspondence $\xi \longmapsto \xi \circ \varphi^*$ defines a linear map from $T_a(M)$ to $T_{\varphi(a)}(N)$.

Proof. $\xi \circ \varphi^*$ is a linear map from $\mathcal{Y}(N)$ to $\mathbb{R}$. Moreover,

$$(\xi \circ \varphi^*)(fg) = \xi(\varphi^* f . \varphi^* g) = \xi(\varphi^* f). g\big(\varphi(a)\big) + f\big(\varphi(a)\big). \xi(\varphi^* g) \tag{4.2}$$

$(f, g \in \mathcal{Y}(N))$ and so $\xi \circ \varphi^* \in T_{\varphi(a)}(N)$. Clearly $\xi \longmapsto \xi \circ \varphi^*$ is linear.

Definition 4.1 (Brickell et al., 1970) : Let $\varphi: M \to N$ be a smooth map and let $a \in M$. The linear map $T_a(M) \to T_{\varphi(a)}(N)$ defined by $\xi \longmapsto \xi \circ \varphi^*$ is called the derivative of φ at a. It is denoted by $(d\varphi)_a$, $((d\varphi)_a \xi)(g) = \xi(\varphi^* g)$, $g \in \mathcal{Y}(N)$, $\xi \in T_a(M)$.

If $\psi: N \to Q$ is a second smooth map, then

$$(d(\psi \circ \varphi))_a = (d\psi)_{\varphi(a)} \circ (d\varphi)_a, \; a \in M.$$

Moreover, for the identity map $\iota: M \to M$, we have

$$(d\iota)_a = \iota_{T_a(M)}, \; a \in M$$

In particular, if $\varphi: M \to N$ is a diffeomorphism, then

$$(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(N) \quad \text{and} \quad (d\varphi^{-1})_{\varphi(a)}: T_{\varphi(a)}(N) \to T_a(M) \quad . \tag{4.3}$$

are inverse linear isomorphisms.

Example 4.1 Let $\varphi: M \to N$ be a smooth map which sends a neighbourhood U of a point $a \in M$ diffeomorphically onto a neighbourhood V of $\varphi(a)$ in N. Then

$$(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(N) \tag{4.4}$$

is a linear isomorphisms.

5. Tangent bundle

In this section we shall discuss properties of smooth map using the concept of tangent bundle. Finally we state our theorem for determining the characteristics of subjective map using the definition of derivative of smooth map. We also give the definition of quotient manifold and sub manifolds by applying the definition of smooth map on manifold.

Definition 5.1 (Brickell et al., 1970) : Let M be a n-manifold. Consider the disjoint union $T_M = \bigcup_{a \in M} T_a(M)$, and let $\pi_M : T_M \to M$ be the projection defined by

$$\pi_M(\xi) = a, \; \xi \in T_a(M).$$

Then we define a manifold structure on T_M so that $\tau_M = (T_M, \pi_M, M, \mathbb{R}^n)$ is a vector bundle over M, whose fibre at a point $a \in M$ is the tangent space $T_a(M)$. Then τ_M is called the tangent bundle of M.

Let $(U_\alpha, u_\alpha, \widehat{U_\alpha})$ be a chart for M and let $j_\alpha: U_\alpha \to M$ be the inclusion map. For each $x \in U_\alpha$ there are linear isomorphisms

$$\lambda_{u_\alpha(x)}: \mathbb{R}^n \xrightarrow{\cong} T_{u_\alpha(x)}(\widehat{U_\alpha})$$
$$(du_\alpha)_x{}^{-1}: T_{u_\alpha(x)}\big(\widehat{U_\alpha}\big) \xrightarrow{\cong} T_x(U_\alpha) \tag{5.1}$$

and

$$(dj_\alpha)_x: T_x(U_\alpha) \xrightarrow{\cong} T_x(M). \tag{5.2}$$

Composing them we obtain a linear isomorphism $\psi_{\alpha,x}: \mathbb{R}^n \xrightarrow{\cong} T_x(M)$.

Finally, let $\{(U_\alpha, u_\alpha)\}$ be an atlas for M. Define maps $\psi_\alpha: U_\alpha \times \mathbb{R} \to T_M$ by

$$\psi_\alpha(x, h) = \psi_{\alpha,x}(h),\ x \in U_\alpha,\ h \in \mathbb{R}^n.$$

If $U_\alpha \cap U_\beta \neq \emptyset$ and $u_{\beta\alpha} = u_\beta \circ u_\alpha{}^{-1}$, the map

$$\psi_{\beta\alpha} = \psi_\beta{}^{-1} \circ \psi_\alpha: U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n$$

given by $\psi_{\beta\alpha}(x, h) = (x, u'_{\beta\alpha}(u_\alpha(x); h))$. Hence it is smooth.

According to the construction of vector bundles, there is a unique vector bundle

$\tau_M = (T_M, \pi_M, M, \mathbb{R}^n)$ for which $\{(U_\alpha, \psi_\alpha)\}$ is a coordinate representation. The fibre of this bundle at $x \in M$ is the tangent space $T_x(M)$.

Evidently this bundle structure is independent of the choice of atlas for M.

Example 5.1 If O is an open subset of vector space E, then the tangent bundle τ_O is isomorphic to the product bundle $O \times E$. In fact, define a map $\lambda: O \times E \to T_O$ by setting

$$\lambda(a, h) = \lambda_a(h),\ \ a \in O,\ h \in E,$$

where λ_a is the canonical linear map. Then λ is a strong bundle isomorphism.

Next, suppose $\varphi: M \to N$ is a smooth map. Then a set map $d\varphi: T_M \to T_M$ is defined by

$$d\varphi(\xi) = (d\varphi)_x \xi, \xi \in T_x(M),\ x \in M.$$

It is called the derivative of φ.

Proposition 5.1 (Ali et al., 2012) : The derivative of a smooth map $\varphi: M \to N$ is a bundle map $d\varphi: \tau_M \to \tau_N$.

Proof. It follows from the definition that $d\varphi$ is fibre preserving and that the restriction of $d\varphi$ to each fibre is linear. To show that $d\varphi$ is smooth we use atlases on M and N to reduce the case $M = \mathbb{R}^n, N = \mathbb{R}^p$. In this case definition 2.5 shows that

$$d\varphi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{R}^p$$

is given by

$$d\varphi(x; h) = (\varphi(x);\ \varphi'(x; h)). \tag{5.3}$$

Hence it is smooth.

Now let $\psi: N \to Q$ be a smooth map into a third manifold. Then

$$d(\psi \circ \varphi) = d\psi \circ d\varphi$$

as follows from the definition. Moreover, the derivative of the identity map $\iota: M \to M$ is the identity map of T_M,

$$d\iota_M = \iota_{T_M}. \tag{5.4}$$

It follows that if $\varphi: M \to N$ and $\psi: M \leftarrow N$ are inverse diffeomorphisms, then $d\varphi$ and $d\psi$ are inverse bundle isomorphisms.

6. Properties of smooth maps on tangent bundle

Let $\varphi: M \to N$ be a smooth map. Then φ is called a local diffeomorphism (respectively an immersion, a submersion) at a point $a \in M$ if the map

$$(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(N) \tag{6.1}$$

is a linear isomorphism (respectively injective, subjective). If φ is a local diffeomorphism (respectively an immersion, a submersion) for all points $a \in M$, it is called a local diffeomorphism (respectively an immersion, a submersion) of M into N.

Theorem 6.1 (Brickell et al., 1970) : Let $\varphi: M \to N$ be a smooth map where dim $M = n$ and dim $N = r$. Let $a \in M$ be given a point. Then

(a) If φ is a local diffeomorphism at a, there are neighbourhoods U of a and V of b such that φ maps U diffeomorphically onto V.

(b) If $(d\varphi)_a$ is injective, there are neighbourhoods U of a and V of b, and W of O in $\mathbb{Q}^{n-r}$, and a diffeomorphism $\psi: U \times W \xrightarrow{\cong} V$ such that $\varphi(x) = \psi(x, 0),\ x \in U$.

(c) If $(d\varphi)_a$ is surjective, there are neighbourhoods U of a and V of b, and W of O in $\mathbb{Q}^{n-r}$, and a diffeomorphism $\psi: U \xrightarrow{\cong} V \times W$ such that $\varphi(x) = \pi_V \psi(x),\ x \in U$, where $\pi_V: V \times W \to V$ is the projection.

Proof. By using charts we may reduce to the case $M = \mathbb{Q}^n,\ N = \mathbb{Q}^r$ in part (a). Then, we are assuming that $\varphi'(a): \mathbb{Q}^n \to \mathbb{Q}^n$ is an isomorphism, and the conclusion in the inverse function theorem.

For part (b), we choose a subspace E of $\mathbb{Q}^r$ such that

$$\text{Im } \varphi'(a) \oplus E = \mathbb{Q}^r. \tag{6.2}$$

and consider the map $\psi: \mathbb{R}^n \times E \to \mathbb{R}^r$ given by

$$\psi(x, y) = \varphi(x) + y,\ x \in \mathbb{Q}^n,\ y \in E.$$

Then

$$\psi^{'}(a,0;h,k) = \varphi^{'}(a;h) + k,\ \ h \in \mathbb{Q}^n,\ \ k \in E \tag{6.3}$$

It follows that $\psi'(a,0)$ is injective and thus an isomorphism $(r = \dim \operatorname{Im} \varphi^{'(a)} + \dim E = n + \dim E)$.

Thus part (a) implies the existence of neighborhoods U of a and V of b, and W of O in E such that $\psi: U \times W \to V$ is a diffeomorphism. Clearly, $\psi(x,0) = \varphi(x)$.

Finally, for part (c), we choose a subspace E of $\mathbb{Q}^r$ such that ker $\varphi'(a) \oplus E = \mathbb{Q}^n$.

Let $\rho: \mathbb{Q}^n \to E$ be the projection induced by this decomposition, and define $\psi: \mathbb{Q}^n \to \mathbb{Q}^r \oplus E$ by

$$\psi(x) = \big(\varphi(x), \rho(x)\big),\ \ a \in \mathbb{Q}^n,\ \ h \in \mathbb{Q}^n. \tag{6.4}$$

It follows easily that $\psi'(a)$ is a linear isomorphism. Hence there are neighbourhoods U of a and V of b, and W of $0 \in E$ such that $\psi: U \to V \times W$ is a diffeomorphism.

Proposition 6.1 : If $\varphi: M \to N$ be a smooth bijective map and if the maps $(d\varphi)_x: T_x(M) \to T_{\varphi(x)}(N)$ are all surjective, then φ is a diffeomorphism.

Proof. Let dim $M = n$ and dim $N = r$. Since $(d\varphi)_x$ is surjective, we have $r \geq n$. Now we show that $r = n$. For every $a \in M$ there exist a neighborhoods U of a and V of b, and W of $0 \in \mathbb{R}^{n-r}$ together with a diffeomorphism $\psi_a: U(a) \times W \xrightarrow{\cong} V$ such that the diagram 1

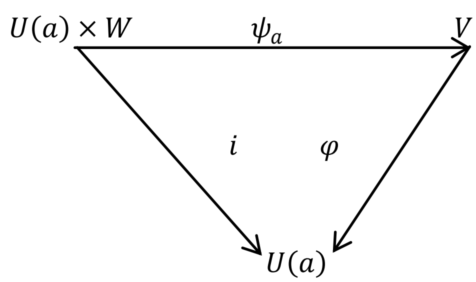

Diagram 1: Composite map

commutes (i denotes the inclusion map opposite 0)

Choose a countable open covering U_i $(i = 1, .., n)$ of M such that each $\overline{U_i}$ is compact and contained in some $U(a_i)$. Since φ is surjective, it follows that $U_i\ \varphi(\ \overline{U_i}) \supset N$.

Now assume that $r > n$. Then the diagram implies that no $\varphi(\bar{U}_i)$ contains an open set. Thus, by the category theorem N could not be Hausdorff. This contradiction shows that $n = r$.

Since $n = r$, φ is a local diffeomorphism. On the other hand, φ is bijective. Since it is a local diffeomorphism and its inverse is smooth. This implies that φ is a diffeomorphism.

7. Quotient manifold

A quotient manifold of a manifold M is a manifold N together with a smooth map $\pi: M \to N$ such that π and each linear map $(d\pi)_x: T_x(M) \to T_{\pi(x)}(N)$ is surjective and

thus dim $M \geq$ dim N.

Proposition 7.1 (Narasimhan. 1968) : Let $\pi: M \to N$ make N into a quotient manifold of M. Assume that $\varphi: M \to Q, \psi: N \to Q$ are maps into a third manifold Q such that the diagram 2

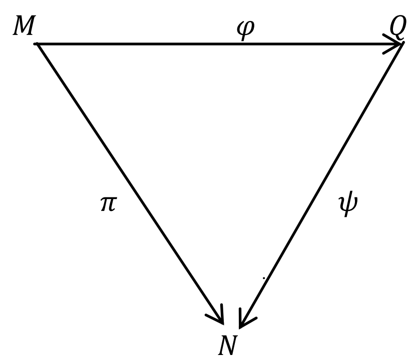

Diagram 2: Smooth map

commutes. Then φ is smooth if and only if ψ is smooth.

8. Sub manifold

Let M be a manifold. An embedded manifold is a fair (N, φ), where N is a second manifold and $\varphi: N \to M$ is a smooth map such that the derivative $d\varphi = T_N \to T_M$ is injective. In particular, since the maps $(d\varphi)_x: T_x(M) \to T_{\varphi(x)}(N)$ are injective, it follows that dim $N \leq$ dim M.

Given an embedded manifold (N, φ), consider the subset $M_1 = \varphi(N)$. φ may be considered as a bijective map

$$\varphi_1: N \to M_1.$$

This bisection defines a smooth structure on M_1, such that φ_1 becomes a diffeomorphism.

Definition 8.1 (Narasimhan. 1968) : A submanifold of a manifold M is an embedded manifold (N, φ) such that $\varphi_1: N \to \varphi(N)$ is a homomorphism, when $\varphi(N)$ is given the topology induced by the topology of M. If N is a subset of M and φ is the inclusion map, we say simply that N is a submanifold of M.

Not every embedded manifold is a submanifold, as the following example shows:

Let M be the 2-torus T^2 and let $N = \mathbb{R}$. Define a map $\varphi: \mathbb{R} \to T^2$ by $\varphi(t) = \pi(t, \lambda t),\ \ t \in \mathbb{R}$,

where λ is an irrational number and $\pi: \mathbb{R}^2 \to T^2$ denotes the projection. Then

$d\varphi: T_{\mathbb{R}} \to T_{T^2}$ is injective and so $(\mathbb{R}, \varphi)$ is an embedded manifold. Since λ is an irrational, $\varphi(\mathbb{R}^+)$ is dense in T^2. In particular there are real numbers $a_i > 0$ such that

$\varphi(a_i) \to \varphi(-1)$. Thus T^2 does not induce the standard topology in $\varphi(\mathbb{R})$.

Proposition 8.1 (Kobayashi et al., 1963) : Let (N, i) be a submanifold of M. Assume that Q is a smooth manifold and

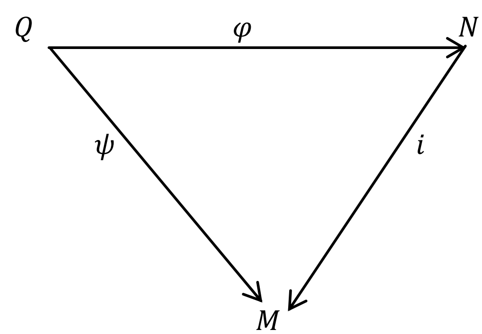

Diagram 3: Commutative map

is a commutative map. Then φ is smooth if and only if ψ is.

Proof. If φ is smooth then clearly so is ψ is. Conversely, assume that ψ is smooth. Fix a point $a \in Q$ and set $b = \psi(a)$. Since di is injective, there are neighborhoods U, V of b in N and M, respectively, and there is a smooth map $\chi: V \to U$ such that $\chi \circ i_U = \iota$.

Since N is a submanifold of M, the map φ is continuous. Hence there is neighborhood W of a such that $\varphi(W) \subset U$. Then $i_U \circ \varphi_W = \psi_W$, where φ_W, ψ_W denote the restrictions of φ, ψ to W. It follows that $\chi \circ \psi_W = \chi \circ i_U \circ \varphi_W = \varphi_W$ and so φ is smooth in W; thus φ is a smooth map.

9. Conclusion

The main objective of this study is finding existence of the diffeomorphism smooth map on manifold. In order to achieve this result we described some propositions and related lemmas to illustrate the theorem. Our result discussed here on the basis of derivative of smooth map on manifolds. Finally we can say that the generalization of the theorem 6.1 can be used in further study on tangent bundle of manifolds for finding the existence of diffeomorphism smooth map.

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